A two-parameter generalization of the complete elliptic integral of the second kind

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Abstract

The double integral

$$E(a,b) = \int_0^\pi dx \int_0^\pi dy \sqrt{1 + a\cos\, x + b\cos\, y} \tag{1}$$

is evaluated in terms of complete elliptic integrals.

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V. Bârsan[1] has recently investigated a double integral equivalent to

$$E(a,b)=\int_0^\pi dx\int_0^\pi dy\sqrt{1+a\cos\ x+b\cos\ y} \tag{1}$$

which, by means of an ingenious procedure, he expressed as a derivative of a hypergeometric function of two variables. In this note we show that (1) can be reduced in a relatively direct manner to a simple combination of complete elliptic integrals. let us assume that $0\le a+b\le 1$ and initially that $Re\ s>0$. Then, in the usual way one has

$$I=\int_0^\pi dx\int_0^\pi dy(1+a\cos\ x+b\cos\ y)^{-s}=$$

$$\frac{1}{\Gamma(s)}\int_0^\infty dst^{s-1}e^{-t}\int_0^\pi dx\int_0^\pi dye^{-at\cos\ x-bt\cos\ y}=$$

$$\frac{\pi^2}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-t}I_0(at)I_0(bt)dt. \tag{2}$$

The latter is a tabulated Laplace transform[2] yielding

$$I=\pi^2F_4(s/2,(s+1)/2;1,1;u(1-v),v(1-u)) \tag{3}$$

with

$$u=\frac{1}{2}[1+a^2-b^2-\sqrt{(1+a^2-b^2)^2-4a^2}]$$

$$v=\frac{1}{2}[1-a^2+b^2-\sqrt{(1-a^2+b^2)^2-4b^2}]. \tag{4}$$

(Note that $u(1-v)=a^2$, $v(1-u)=b^2$). Now, by analytic continuation we can take $s=-1/2$. Next, by L. Slater's reduction formula[3], one has

$$E(a,b)=\pi^2[\ {}_2F_1(-1/4,1/4;1;u)\ {}_2F_1(-1/4,1/4;1;v)+$$

$$\frac{1}{16}uv\ {}_2F_1(3/4,5/4;2;u)\ {}_2F_1(3/4,5/4;2;v)]. \tag{5}$$

Finally, since[4]

$${}_2F_1(-1/4,1/4;1;z^2)=\frac{2}{\pi}\sqrt{1+z}\mathbf{E}(k)$$

$${}_2F_1(3/4,5/4;2;z^2)=\frac{8}{\pi z^2\sqrt{1+z}}[\mathbf{K}(k)-(1+z)\mathbf{E}(k)] \tag{6}$$

$$k=\sqrt{\frac{2z}{1+z}}$$

we have the desired expression

$$\frac{1}{4}E(a,b) = 2\sqrt{(1+\sqrt{u})(1+\sqrt{v})}\mathbf{E}[k(\sqrt{u})]\mathbf{E}[k(\sqrt{v})]+$$

$$\frac{\mathbf{K}[k(\sqrt{u})]\mathbf{K}[k(\sqrt{v})]}{\sqrt{(1+\sqrt{u})(1+\sqrt{v})}} - \sqrt{\frac{1+\sqrt{u}}{1+\sqrt{v}}}\mathbf{E}[k(\sqrt{u})]\mathbf{K}[k(\sqrt{v})]-$$

$$\sqrt{\frac{1+\sqrt{v}}{1+\sqrt{u}}}\mathbf{E}[k(\sqrt{v})]\mathbf{K}[k(\sqrt{u})] \tag{7}$$

For the case $a = b$ (3) and (7) simplify to

$$\frac{1}{4}\int_0^\pi \int_0^\pi \sqrt{1+a(\cos\ x+\cos\ y)}dxdy = \frac{\pi^2}{4}\ {}_3F_2(-1/4, 1/4, 1/2; 1, 1; 4a^2) =$$

$$2(1+\sqrt{u})\mathbf{E}^2(k) + (1+\sqrt{u})^{-1}\mathbf{K}^2(k) - 2\mathbf{E}(k)\mathbf{K}(k), \tag{8}$$

where $u = (1-\sqrt{1-4a^2})/2$, $k = \sqrt{2\sqrt{u}/(1+\sqrt{u})}$.

References

[1] V. Bârsan, arXiv:0708.2325v1

[2]. A.P.Prudnikov, Yu. A. Brychkov and O.I. Marichev, *Integrals and Series, Vol.2.* Nauka, Moscow 1981.

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